A mapping construction for the *q*-deformed $so(3) \subset u(3)$ embedding

1999 J. Phys. A: Math. Gen. 32 2403

(http://iopscience.iop.org/0305-4470/32/12/013)

J. Phys. A: Math. Gen. **32** (1999) 2403–2413. Printed in the UK PII: S0305-4470(99)97458-9

A mapping construction for the q-deformed $so(3) \subset u(3)$ embedding

A I Georgieva†, J D Goleminov†, M I Ivanov† and H B Geyer‡

† Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia 1784, Bulgaria

‡ Institute of Theoretical Physics, University of Stellenbosch, 7599 Stellenbosch, South Africa

Received 8 September 1998

Abstract. Anticipating subsequent applications in nuclear structure theory, a first construction of a Dyson mapping for a q-deformed $u(3)$ algebra, relevant to this field, is presented. To achieve this, a q-deformed $sp(4, R)$ algebra is initially considered, realized in terms of tensor operators with respect to the standard $su_q(2)$ and containing a q-deformed $so(3)$ angular momentum algebra. The desired mapping is then realized in terms of two boson-type conjugated tensor operators of first rank. A key problem is to determine the commutation relations between them. Our construction is based on the requirement that subsets of the commutation relations of the original $so(3)$ algebra is preserved. As a result the images of the $so(3)$-subalgebra of $sp(4, R)$ close the same commutation relations as the initial subalgebra of the angular momentum. In addition a q-deformed $u(3)$ algebra, containing the $so(3)$-subalgebra of the images, is obtained. Its generators are the q-deformed components of a quadrupole operator, together with the images of the $so(3)$-subalgebra. In the limiting case $q \to 1$ the reduction $su(3) \supset so(3)$, crucial to nuclear structure physics, is recovered.

1. Introduction

In the last decade interest in deformed algebraic structures, introduced some time ago, has been re-established and much work has been done both in developing the mathematical theory of quantum algebras and at the same time extending their physical applications [1]. Some applications, aiming to explore the possible role of the q-deformation parameter in the theory of nuclear collective structure have, for example, been reported at the level of $su_q(2)$ [2–4]. Nevertheless, more general applications in this field are still restricted, a situation which may be addressed by further exploring q-deformed group theoretical structures and methods, in particular linked to the q-deformed extensions of algebraic models associated with nuclear collective motion [5, 6].

A case in point concerns the reduction of the $su(3)$ algebra, containing the components $Q^2_m (m = 0, \pm 1, \pm 2)$ of the algebraic quadrupole operator, to the $so(3)$ algebra of the angular momentum $L^1_m (m = 0, \pm 1)$. This is a basic and crucial element, common to nuclear collective models which exploit the important quadrupole degree of freedom in conjunction with a classification scheme which utilizes angular momentum, starting with the Elliott $SU(3)$ model [7] and also using, for example, in the pseudo-$SU(3)$ model [8], the symplectic collective model [9] and the interacting boson model (IBM) [10]. However, this reduction is a complicated, and not completely resolved problem in the q-deformed extension, into which much effort has recently been put [11, 12].

0305-4470/99/122403+11$19.50

In [11] the construction of $su_q(3)$ is approached by generating an $so_q(3)$-algebra, isomorphic to the standard $su_q(2)$, in terms of three-dimensional (3D) q-oscillators. By construction, this $su_q(2)$ is then a subalgebra of the $su_q(3)$ in the Chevalley basis, but its generators do not form a q-deformed tensor of first rank. Furthermore, this approach does not facilitate the construction of the q-analogue of the physically important quadrupole operator. In our construction later we succeed in addressing both these issues.

What is obtained in [11] is a 3D representation of $su_q(2)$ which is further used by Quesne [12] to define q-bosonic operators which transform as q-tensor operators (q-vectors) under this $su_q(2)$. Their coupled commutators are written down in [12] in analogy with the classical case for the semidirect sum of the Heisenberg–Weyl Lie algebra $w(3)$ and $so(3)$. A Dyson representation in terms of the standard q-bosonic oscillators is then obtained. By irreducible tensor coupling a scalar, a vector and a rank two tensor are proven to generate a q-deformation of a quadratic $u(3)$ which, however, is not a q-analogue of the Elliott $u(3)$ algebra, as obtained by us.

Our construction is based, among other considerations, on the observation that a very natural way to obtain the embedding of the angular momentum algebra is to consider tensor operators with respect to it and then to generate the higher rank algebras in terms of these operators. This is a procedure used in most of the well known algebraic models. In order to extend this technique one needs a well developed theory of the angular momentum in the q-deformed case, but the Wigner–Racah algebra for $SU_q(2)$ is already well developed and so we will use the definitions and results of [13].

Based on this idea a q-deformed $sp(4, R)$ algebra is generated in [14, 15] by the possible tensor products of the two fundamental $SU_q(2)$ spinors. In the classical case this algebra [16] is of physical interest by itself and is also easily generalized to the higher rank cases—$sp(2n, R)$, $n = 3, 4, \ldots$. In general, the group $Sp(2n)$ can be used to describe pairing correlations in systems containing different kinds of particles [17], while the non-compact version of this group is applied in the description of collective vibrational excitations of a system of particles moving in an n-dimensional harmonic oscillator potential. With similar applications in mind in the q-deformed case we emphasise that a natural procedure entails embedding into a q-deformed $sp(4, R)$ algebra, by construction, a q-deformed $so(3)$-subalgebra, the latter being generated by the components of a first-rank tensor L^1_m $(m = 0, \pm 1)$ which can be interpreted as an angular momentum operator.

A further consideration is that the symplectic algebras are particularly convenient for boson mapping, a situation often exploited in nuclear structure physics. Originally the mapping methods [18] were formulated and motivated from the point of view of replacing fermion degrees of freedom directly with exact boson degrees of freedom. This mapping of fermion pairs provides a certain microscopic justification for various boson models of nuclear structure. Further development of these methods also points to the efficiency and usefulness of a group theoretical interpretation of the formalism and facilitated their generalization to other systems which have a definite algebraic structure, such as boson pairs and spherical tensors. Mappings of this kind have also been used to obtain the relationship between the different boson models [19].

This mapping procedure generally leads to a larger space for the boson images of the initial algebra. Reducing this larger space to the required dimension of the initial one is formulated as the identification of non-physical or spurious states [20]. A solution of this problem can be obtained by means of a pure group-theoretical analysis, both in the case of the mapping of fermion (compact) [21] or boson (non-compact) [19] pairs (algebras). This procedure is actually equivalent to obtaining the embedding of the image of the initial algebra in the larger space of the images.

Our idea is to use a similar procedure in the q-deformed case in order to obtain through the mapping of the $sp(4, R)$ algebra the embedding of the image of its $so(3)$ subalgebra in the larger space of the images, which is proven to be spanned by the generators of a $u(3)$ q-deformed algebra. For this purpose, in analogy with the classical case, a Dyson mapping of the q-deformed $sp(4, R)$ obtained in [15] is considered.

The paper is organized as follows: we first present notation, definitions and a short review of the q-deformed $sp(4, R)$ algebra. For this algebra the mapping procedure is then given step by step, using, where possible, analogies with the classical Dyson mapping. The resulting algebraic structure associated with the image of the compact subalgebra $so(3)$ can then be utilized to construct images for q-deformed quadrupole operators, finally leading to a $u(3)$ realization and the important embedding $u(3) \subset so(3)$ for q-algebras.

2. The initial q-deformed $sp(4, R)$

In [14, 15] a q-deformation of the physically interesting $sp(4, R)$ algebra is generated in terms of irreducible q-tensor operators with respect to the oscillator representation of the standard $su_q(2)$ algebra, defined through the commutation relations of its generators $J_\pm$ and J_0

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_0] \tag{1}$$

where $[X] \equiv (q^X - q^{-X})/(q - q^{-1})$.

As in [13] an irreducible tensor operator (ITO) T^l of rank l is defined through the commutation relations of its components ${}_qT^l_m$ $(m = -l, -l+1, \ldots, l)$ with the $su_q(2)$ generators

$$[J_0, {}_qT^l_m] = m\, {}_qT^l_m \tag{2}$$

$$[J_\pm, {}_qT^l_m]_{q^m} = \sqrt{[l \mp m][l \pm m + 1]}_q\, T^l_{m\pm 1} q^{-J_0} \tag{3}$$

where

$$[A, B]_{q^m} = AB - q^m BA \tag{4}$$

and m is called the degree of the q-commutator. The oscillator representation [22] of the algebras considered is given in terms of annihilation a_i, creation $a_i^\dagger$ (with $(a_i^\dagger)^\dagger = a_i$) and number operators N_i of q-bosons $(i = 1, 2, \ldots, r)$ which satisfy

$$a_i a_i^\dagger - q^1 a_i^\dagger a_i = q^{-N_i} \tag{5}$$

$$a_i a_i^\dagger - q^{-1} a_i^\dagger a_i = q^{N_i} \tag{6}$$

$$[N_i, a_i] = -a_i \qquad [N_i, a_i^\dagger] = a_i^\dagger. \tag{7}$$

The operators $a_i, a_i^\dagger$ and N_i for $i = 1, 2$ do not form an ITO [1] with respect to the boson oscillator representation of the $su_q(2)$ algebra

$$J_+ = a_1^\dagger a_2 \qquad J_- = a_2^\dagger a_1 \qquad J_0 = \frac{N_1 - N_2}{2}. \tag{8}$$

However, the following modification of the creation and annihilation operators [1, 15]

$$t_{1/2,1/2} = a_1^\dagger q^{N_2/2} \qquad t_{1/2,-1/2} = a_2^\dagger q^{-N_1/2} \tag{9}$$

$$\tilde{t}_{1/2,1/2} = q^{-1/2} a_2 q^{-N_1/2} \qquad \tilde{t}_{1/2,-1/2} = -q^{1/2} a_1 q^{N_2/2} \tag{10}$$

defines the components of two tensors of rank 1/2 (spinors) with respect to the boson [1] representation of the $su_q(2)$ algebra defined by the generators (8). We observe that in the q-deformed case the construction of boson $su_q(2)$ tensor operators is more complicated than in the classical case.

Working in terms of ITOs gives us the possibility to use the tensor product with respect to the $su_q(2)$, defined in accordance with standard comultiplication rules for this algebra [13]. The tensor product of two ITOs is defined as in the classical case in the following way

$$({}_qT^{(l)} \otimes {}_qQ^{(k)})^{(r)} := \sum_{m_1 m_2} {}_qC^{rm}_{lm_1km_2}\, {}_qQ^{(k)}_{m_2}\, {}_qT^{(l)}_{m_1} \tag{11}$$

where the following restrictions hold

$$r = |k-l|, |k-l+1|, \ldots, k+l \qquad m = m_1 + m_2 = -r, -r+1, \ldots, r$$
$$m_1 = -l, -l{+}1, \ldots, l \qquad m_2 = -k, -k+1, \ldots, k.$$

The ${}_qC^{rm}_{lm_1km_2}$ are Clebsch–Gordan coefficients (CGCs) for the $su_q(2)$ Wigner–Racah algebra, obtained in [13].

From the two fundamental $su_q(2)$ spinors (9) and (10) the following tensor products can be constructed by means of the definition (11)

$$\{t_{1/2} \otimes t_{1/2}\}^l_m := T^l_m \qquad l = 1, \quad m = 0, \pm 1 \tag{12}$$
$$\{\tilde{t}_{1/2} \otimes \tilde{t}_{1/2}\}^l_m := \tilde{T}^l_m \qquad l = 1, \quad m = 0, \pm 1 \tag{13}$$
$$\{\tilde{t}_{1/2} \otimes t_{1/2}\}^l_m := L^l_m \qquad l = 0, 1, \quad m = -l, -l{+}1, \ldots, l. \tag{14}$$

The explicit form of the operators T^1_m, $\tilde{T}^1_m$ and L^l_m in terms of boson annihilation and creation operators a_i and $a_i^\dagger$, $i = 1, 2$, is given in [14]. Important properties of the tensor operators obtained in this way are their conjugation relations

$$(T^1_m)^\ddagger = (-1)^{1-m} q^{-m} \tilde{T}^1_{-m} \qquad (\tilde{T}^1_m)^\ddagger = (-1)^{1-m} q^{-m} T^1_{-m} \tag{15}$$
$$(L^1_m)^\ddagger = (-1)^{-m} q^{-m} L^1_{-m}. \tag{16}$$

Using equations (5), (6) and (7), the commutation relations between the operators T^1_m, $\tilde{T}^1_m$; $m = 0, \pm 1$ and L^l_m; $l = 0, 1$; $m = -l, -l{+}1, \ldots, l$ are calculated in [14]. In so doing the q-boson commutation relations are applied according to the convention that equation (5) applies to $i = 1$, while equation (6) applies to $i = 2$. This convention complies with the tensor nature of the spinors (9) and (10) and the tensor products (12), (13) and (14).

Later we introduce and discuss these commutation relations, from a point of view which is more appropriate for the purpose of obtaining a q-boson realization of the q-deformed $sp(4, R)$, finally to be extended to $u(3)$.

First, we focus on the components of the operator L^1_m which satisfy

$$[L^1_{m_1}, L^1_{m_2}] = m_1 q^{-(m_1+m_2)} \sqrt{[2]}\, L^1_{m_1+m_2} q^{-2J_0} \qquad m_1 \neq 0. \tag{17}$$

On the right-hand side of equation (17) we see that, in addition to the expected appropriate component $L^1_{m_1+m_2}$, the operator q^{-2J_0}, which in the limit $q \to 1$ tends to one, also appears. This is a direct consequence of the deformation limit and can be traced explicitly to the q-boson commutator convention mentioned earlier.

The operator q^{-2J_0} obviously commutes with the tensor operators (12), (13) and (14), according to their tensorial properties (2), in the following way

$$[q^{-2J_0}, K^l_m]_{q^{-2m}} = 0 \tag{18}$$

where K^l_m represents any of the ten components of the operators L^l_m, T^1_m and $\tilde{T}^1_m$, $l = 0, 1$; $m = 0, \pm 1$. It is simple to transform away q^{-2J_0} on the right-hand side of equation (17) if we rescale, as in [15], each component of the generators to

$$\widehat{K^l_m} = \sqrt{[1 + \delta_{m,0}]}\, K^l_m q^{-2J_0} \qquad l = 0, 1, \quad m = 0, \pm 1. \tag{19}$$

This rescaling only changes the degree of the q-commutators, as can, for example, be seen explicitly in the modification of the commutation relations (17)

$$[\widehat{L^1_{m_1}}, \widehat{L^1_{m_2}}]_{q^{2(m_2-m_1)}} = [m_1 - m_2]q^{(m_2-m_1)}\widehat{L}^1_{m_1+m_2}. \tag{20}$$

The operators $\widehat{L^1_m}$ can thus be interpreted as generators of a q-deformed $so(3)$ algebra, isomorphic to a q-deformed $su(2)$. These operators and the algebra generated by them is of central importance for the physical applications we have in mind and their mapping is our main focus next.

In the following developments results will be expressed in terms of the tensor operators (12), (13) and (14), which will be used in the mapping, having in mind that the rescaling (19) can and should always be performed after the calculation of commutation relations in order to transform away the operator q^{-2J_0}.

The scalar operator

$$L^0_0 = \frac{[N]}{\sqrt{[2]}} = \frac{1}{\sqrt{[2]}}\{a_1^\dagger a_1 q^{N_2} + a_2^\dagger a_2 q^{-N_1}\} = \frac{1}{\sqrt{[2]}}\{[N_1]q^{N_2} + [N_2]q^{-N_1}\} \tag{21}$$

commutes with all components of the first-rank tensor L^1_m

$$[L^1_m, L^0_0] = 0 \qquad m = -1, 0, 1. \tag{22}$$

Hence (21) is an invariant of the subalgebra defined in (17). The combinations $\mathcal{N}_\mu = L^1_0 + \mu q^\mu L^0_0$ given explicitly for $\mu = \pm 1$ as

$$\mathcal{N}_1 = L^1_0 + qL^0_0 = \sqrt{[2]}[N_2]q^{-N_1} \qquad \mathcal{N}_{-1} = \quad L^1_0 - q^{-1}L^0_0 = -\sqrt{[2]}[N_1]q^{N_2} \tag{23}$$

play the role of q-deformations of operators for the number of bosons of each kind. The operators (23) can thus be considered as Cartan generators of a q-deformed tensor $u(2)$ algebra with the operators L^1_1 and L^1_{-1} as raising and lowering generators. In terms of the tensor operators L^1_m and L^0_0 the reduction of a q-deformed tensor $u(2) \supset su(2) \otimes u(1) \sim so(3) \oplus o(2)$ is so realized.

The pair operators T^1_m, $\tilde{T}^1_m$ fulfil the following q-commutation relations

$$[T^1_{m_1}, T^1_{m_2}]_{q^{2(m_1-m_2)}} = 0' \qquad [\tilde{T}^1_{m_1}, \tilde{T}^1_{m_2}]_{q^{2(m_1-m_2)}} = 0. \tag{24}$$

The commutation relations between the T^1_m and $\tilde{T}^1_m$ close in terms of the respective components of the tensor operators (14). The subset with $m_1 + m_2 \neq 0$ can be presented in a unified way as

$$[T^1_{m_1}, \tilde{T}^1_{m_2}]_{q^{2(m_2-m_1)}} = -q^{-2m_1}\sqrt{[2]}[2(m_2 - m_1)]L^1_{m_1+m_2}q^{-2J_0} \tag{25}$$

while for the case of $m_1 + m_2 = 0$ we obtain

$$\begin{aligned}
&[T^1_1, \tilde{T}^1_{-1}]_{q^{-4}} = -q^{-3}[2]q^{-4J_0} + q^{-1}[2]\sqrt{[2]}\mathcal{N}_{-1}q^{-2J_0} \\
&[T^1_0, \tilde{T}^1_0] = [2]q^{-4J_0} + \sqrt{[2]}(q^{-1}\mathcal{N}_1 - q\mathcal{N}_{-1})q^{-2J_0} \\
&[T^1_{-1}, \tilde{T}^1_1]_{q^4} = -q^3[2]q^{-4J_0} - q[2]\sqrt{[2]}\mathcal{N}_1 q^{-2J_0}.
\end{aligned} \tag{26}$$

In [14] the commutation relations of the operator L^1_m, $m = 0, \pm 1$ with T^1_m and $\tilde{T}^1_m$ are also given. Taken together with the commutators (26) it can then be shown that in the limit $q \to 1$ the correct commutation relations for an $sp(4, R) \sim o(3, 2) \supset so(3) \oplus o(2)$ algebra [17] are closed. This identification permits us, after the rescaling (19), to define by means of an $su_q(2)$ ITO a q-deformation of the physically interesting $sp(4, R)$ algebra.

3. The mapping procedure

For the tensor q-deformed $sp(4, R)$ algebra constructed in [14, 15] and outlined in the previous section, a generalized Gauss decomposition [19] $\mathcal{G} = g_- + h + g_+$, is obtained in a natural way. This follows as the compact subalgebra h is generated by the operators $\widehat{L^l_m}$ $(l = 0, 1; m = 0, \pm 1)$, while g_+ and g_- are two nilpotent (24) subalgebras containing the components of the two conjugated first-rank tensors $\widehat{T^l_m}$ and $\widehat{\tilde{T}^1_m}$, respectively.

We can now consider the Dyson mapping of the tensor q-deformed $sp(4, R)$, by applying in analogy with the classical case the procedure [23], developed for symplectic algebras spanned over spherical tensors.

The first step in the mapping is the definition of the images of one of the pair raising or lowering subalgebras g_+ or g_-. In the case considered we first determine the image of $\tilde{T}^1_m$ as

$$\tilde{T}^1_m \to \tilde{b}_m \qquad m = 0, \pm 1. \tag{27}$$

The 'annihilation' operators $\tilde{b}_m$ introduced in this way are assumed to be components of a first-rank tensor with respect to $su_q(2)(1)$. Another first-rank tensor $b^\dagger_m$ is introduced as the one conjugated to $\tilde{b}^1_m$ in the same way as T^1_m and $\tilde{T}^1_m$ are related in expression (15)

$$(\tilde{b}_m)^\ddagger := (-1)^{(1-m)} q^{-m} b^\dagger_{-m} \qquad (b^\dagger_m)^\ddagger := (-1)^{(1-m)} q^{-m} \tilde{b}_{-m}. \tag{28}$$

The two ITOs with respect to the $su_q(2)$ q-vectors $b^\dagger_m$ and $\tilde{b}_m$ play the role of mapping tools by means of which we have to construct the images of our initial algebra, in this particular case of the compact subalgebra h.

As the operators $\tilde{b}_m$ and $b^\dagger_m$ are by definition components of two first-rank $su_q(2)$ tensors, the images of

$$L^1_m \to I^1_m \qquad m = -1, 0, 1 \tag{29}$$

can be viewed as components of the first-rank tensor product

$$I^1_m := (\tilde{b} \otimes b^\dagger)^1_m = \sum_{m_1, m_2} {}_q C^{1m=m_1+m_2}_{1m_1 1m_2} b^\dagger_{m_2} \tilde{b}_{m_1}. \tag{30}$$

After calculating the q-deformed CGCs, the explicit representation for the components is found as

$$\begin{aligned} I^1_{-1} &= \sqrt{\frac{[2]}{[4]}} \{ q b^\dagger_{-1} \tilde{b}_0 - q^{-1} b^\dagger_0 \tilde{b}_{-1} \} \\ I^1_0 &= \sqrt{\frac{[2]}{[4]}} \{ b^\dagger_{-1} \tilde{b}_1 - b^\dagger_1 \tilde{b}_{-1} + (q - q^{-1}) b^\dagger_0 \tilde{b}_0 \} \\ I^1_1 &= \sqrt{\frac{[2]}{[4]}} \{ q b^\dagger_0 \tilde{b}_1 - q^{-1} b^\dagger_1 \tilde{b}_0 \}. \end{aligned} \tag{31}$$

By means of the conjugation rules (28) we may check the conjugation of the operators (31) and obtain for the images the same result (16) as for the initial operators

$$(I^1_m)^\ddagger = (-1)^{-m} q^{-m} I^1_{-m}. \tag{32}$$

A complete justification for our definition (29) of the images of the operators L^1_m requires that we prove that the images close the same set of commutation relations (17) as the initial operators.

This clearly requires that we know what kind of algebra the basic elements used in the construction of the mapping, i.e. $b^\dagger_m$ and $\tilde{b}_m$, close by themselves. In the classical case this is a

Weyl algebra $W(3)$, defined by the standard boson commutation relations of the creation and annihilation operators. In the q-deformed case the relation between boson operators and ITOs with respect to $su_q(2)$ is not so easily obtained, so that it is not immediately obvious which commutation relations are to be used between the q-vector operators. Our idea is to use, in analogy with the classical case but appropriately modified, the commutation relations of the original tensor operators (12) and (13).

With this in mind, we first define the following set of q-commutation relations

$$\begin{aligned} &[\tilde{b}_m, \tilde{b}_{m'}]_{q^\lambda} := [\tilde{T}^1_m, \tilde{T}^1_{m'}]_{q^\lambda} = 0 \\ &[b^\dagger_m, b^\dagger_{m'}]_{q^\lambda} := [T^1_m, T^1_{m'}]_{q^\lambda} = 0 \qquad m, m' = -1, 0, 1 \end{aligned} \tag{33}$$

to be equivalent to the commutation relations of the respective g_- and g_+ generators given by (24). This fixes $\lambda = 2(m - m')$.

The set of $[b^\dagger_m, b_{m'}]_{q^\mu}$ commutators is now separated and treated in two parts.

- The first subset has $m + m' \neq 0$, for which the corresponding q-commutators $[T^1_m, \tilde{T}^1_{m'}]_{q^a}$ give terms proportional to the operators L^1_m, $m = \pm 1$ as in (25).

For this set we retain the type of q-commutator, but neglect the resulting generator, by defining

$$[b^\dagger_m, \tilde{b}_{m'}]_{q^\mu} := 0. \tag{34}$$

The value $\mu = -\lambda = 2(m' - m)$ is hereby determined.

- The second subset consists of combinations of indices with $m + m' = 0$ as in (26) where the results for the generator commutators can all be separated into a first free term together with an L-dependent term.

Taking this structure as a guideline for the pairs of indices $(m, m') = (1, -1)$, $(0, 0)$ and $(-1, 1)$, we again retain the corresponding degree of the q-commutator, neglect the L-dependent term and hence assume the following q-commutators

$$\begin{aligned} &[b^\dagger_{-1}, \tilde{b}_1]_{q^4} := -\alpha q^4 q^{-\beta J_0} \\ &[b^\dagger_0, \tilde{b}_0] := \gamma q^{-\beta J_0} \\ &[b^\dagger_1, \tilde{b}_{-1}]_{q^{-4}} := \delta q^{-\beta J_0} \end{aligned} \tag{35}$$

in which the parameters α, γ and δ, can be q-numbers, while β is a real number.

Our mapping construction now proceeds in analogy with the commutator mapping methods [23] in the classical case. The values of the parameters α, γ, δ and β in (35) are accordingly evaluated by imposing the basic condition that the commutation relations between the images $[I^1_m, I^1_{m'}]$ should be the same as the commutation relations between the initial operators $[L^1_m, L^1_{m'}]$ in (17):

$$[L^1_m, L^1_{m'}] \to [I^1_m, I^1_{m'}]. \tag{36}$$

We now substitute expressions (31) for the operators I^1_m, $m = 0, \pm 1$, in these commutators and use the commutation relations (33), (34) and (35). Taking care that operator products are consistently treated in such a way that a non-trivial deformation is retained, we obtain simple equations for the parameters, which have the following solutions

$$\alpha = q^{-1}\sqrt{[4]} \qquad \beta = 2 \qquad \gamma = \sqrt{[4]} \qquad \delta = -q^{-3}\sqrt{[4]}. \tag{37}$$

Introducing these values of the parameters α, β, γ and δ into the q-commutators (35), we finally infer the last set of commutation relations for the operators $b_m^\dagger, \tilde{b}_m$, $m = 0, \pm 1$, which can now be jointly expressed as

$$[b_m^\dagger, \tilde{b}_{m'}]_{q^\mu} = (-1)^{2m'-m} q^{2m'-m} \delta_{m,-m'} \sqrt{[4]} q^{-2J_0} \tag{38}$$

with $\mu = 2(m' - m)$.

Here it must be noted that by comparing the right-hand side of (38) with the free term of (26) only the J_0 degree of the operator q and the q-number are changed. Also we do not know the representation of the operator J_0 in terms of the operators $b_m^\dagger, \tilde{b}_m$, but we assume that their action on the tensor operators (2) and (3) is preserved, for example

$$q^{J_0} b_m^\dagger = q^m b_m^\dagger q^{J_0} \qquad q^{J_0} \tilde{b}_m = q^m \tilde{b}_m q^{J_0}. \tag{39}$$

The set of commutation relations (33), (34) and (38) are evaluated in such a way that the commutation relations between the images I_m^1 are the same as the commutation relations between the initial operators L_m^1 (17). In order to remove the additional operator q^{-2J_0} the same rescaling (19) of the components of the image operator I_m^1 can be introduced

$$\widehat{I_m^1} = \sqrt{[1 + \delta_{m,0}]} I_m^1 q^{2J_0} \qquad m = 0, \pm 1. \tag{40}$$

Hence as a result of mapping and rescaling (40) the operators $\widehat{I_m^1}$ generate a q-deformed $so(3)$ algebra with q-commutation relations

$$[\widehat{I_{m_1}^1}, \widehat{I_{m_2}^1}]_{q^{2(m_2-m_1)}} = [m_1 - m_2] q^{(m_2-m_1)} \widehat{I_{m_1+m_2}^1}. \tag{41}$$

4. The q-deformed $su(3)$ algebra

The construction of the previous section expresses the image of the subalgebra h of the q-deformed tensor $sp(4, R)$ algebra in terms of the two conjugated ITOs $b_m^\dagger$ and $\tilde{b}_m$. As these operators are, by definition, first-rank tensors with respect to $su_q(2)$, we can also construct, according to the rules for tensor products (11), the operators

$$Q_m^2 := (b^\dagger \otimes \tilde{b})_m^2 = \sum_{m_1 m_2} C_{1m_1 1m_2}^{2m} b_{m_2}^\dagger \tilde{b}_{m_1} \tag{42}$$

with $m = 0, \pm 1, \pm 2$, $m_1, m_2 = -1, 0, 1$ and $m = m_1 + m_2$, as well as

$$S_0^0 := (b^\dagger \otimes \tilde{b})_0^0 = \sum_{m_1 m_2} C_{1m_1 1m_2}^{00} b_{m_2}^\dagger \tilde{b}_{m_1}. \tag{43}$$

The components of the second-rank tensor operator, expressed in terms of the mapping operators $b_m^\dagger, \tilde{b}_m$ are

$$\begin{aligned} &Q_2^2 = b_1^\dagger \tilde{b}_1 \qquad Q_{-2}^2 = b_{-1}^\dagger \tilde{b}_{-1} \\ &Q_1^2 = \sqrt{\frac{[2]}{[4]}} \{q^{-1} b_0^\dagger \tilde{b}_1 + q b_1^\dagger \tilde{b}_0\} \qquad Q_{-1}^2 = \sqrt{\frac{[2]}{[4]}} \{q b_0^\dagger \tilde{b}_{-1} + q^{-1} b_{-1}^\dagger \tilde{b}_0\} \\ &Q_0^2 = \sqrt{\frac{[2]}{[3][4]}} \{q^2 b_1^\dagger \tilde{b}_{-1} + q^{-2} b_{-1}^\dagger \tilde{b}_1 + [2] b_0^\dagger \tilde{b}_0\}. \end{aligned} \tag{44}$$

We can interpret this ITO of second rank as a quadrupole operator. Such an interpretation is justified first by the commutation relation of the component of this operator with the operators I_m^1 (31). They are calculated with the help of the commutation relations obtained in the previous section. We give here the results for the rescaled operators $\widehat{I_m^1}$ in (40) and $\widehat{Q_m^l} = Q_m^l q^{-2J_0}$ $(m = 0, \pm 1, \pm 2)$, respectively

$$[\widehat{I_{m_1}^1}, \widehat{Q_{m_2}^2}]_{q^{2(m_2-m_1)}} = q^{(m_2-m_1)} \mathcal{F}_{m_1,m_2} \widehat{Q^2}_{m_1+m_2} \tag{45}$$

where

$$\mathcal{F}_{\pm1,m_2} = \pm\sqrt{[2 \mp m_2][2 \pm m_2 + 1]}$$
$$\mathcal{F}_{0,m_2} = -[m_2]. \tag{46}$$

These commutators can be used as defining relations for the components of a tensor operator with respect to the $su(2) \sim so(3)$ q-deformed algebra, generated by the operators $\widehat{I^1_m}$, $m = 0, \pm1$. The transition to the classical limit for $q \to 1$ reproduces the results

$$[\dot{L}_{M_1}, \dot{Q}_{M_2}] = -\sqrt{6}C^{2M_1+M_2}_{1M_12M_2}\dot{Q}_{M_1+M_2} \qquad M_1 = 0, \pm1, M_2 = 0, \pm1, \pm2 \tag{47}$$

for the 'algebraic' quadrupole operator $\dot{Q}^2_M$ and the real angular momentum operator $\dot{L}^1_M$, as $SU(3)$ generators [7, 8].

Furthermore, the commutation relations between the components of the operator $\widehat{Q^2_m}$ $(m = 0, \pm1, \pm2)$ read

$$[\widehat{Q^2_{-2}}, \widehat{Q^2_{-1}}]_{q^2} = 0 \qquad [\widehat{Q^2_1}, \widehat{Q^2_2}]_{q^2} = 0$$
$$[\widehat{Q^2_{-1}}, \widehat{Q^2_0}]_{q^2} = -q\sqrt{[2][3]}\{\widehat{I^1_{-1}} - (q - q^{-1})[2]\widehat{Q^2_{-1}}\}$$
$$[\widehat{Q^2_0}, \widehat{Q^2_1}]_{q^2} = -q\sqrt{[2][3]}\{\widehat{I^1_1} - (q - q^{-1})[2]\widehat{Q^2_1}\}$$
$$[\widehat{Q^2_{-2}}, \widehat{Q^2_0}]_{q^4} = q^2(q - q^{-1})\sqrt{[2][3]}\widehat{Q^2_{-2}} \qquad [\widehat{Q^2_0}, \widehat{Q^2_2}]_{q^4} = q^2(q - q^{-1})\sqrt{[2][3]}\widehat{Q^2_2}$$
$$[\widehat{Q^2_{-2}}, \widehat{Q^2_1}]_{q^6} = q^3\widehat{I^1_{-1}} \qquad [\widehat{Q^2_{-1}}, \widehat{Q^2_2}]_{q^6} = q^3\sqrt{[4]}\widehat{I^1_1}$$
$$[\widehat{Q^2_{-2}}, \widehat{Q^2_2}]_{q^8} = q^4\{[2]^2\widehat{I^1_0} - (q - q^{-1})\sqrt{[2][3]}\widehat{Q^2_0}\}$$
$$[\widehat{Q^2_{-1}}, \widehat{Q^2_1}]_{q^4} = -q^2[2]\{\widehat{I^1_0} - (q - q^{-1})\sqrt{[2][3]}\widehat{Q^2_0}\}. \tag{48}$$

Analysing them, it should be noted that they are q-commutators, which give the correct components of the operator $\widehat{I^1_m}$, according to the rule

$$[\dot{Q}_{M_1}, \dot{Q}_{M_2}] = 3\sqrt{10}C^{1M_1+M_2}_{2M_12M_2}\dot{L}_{M_1+M_2} \qquad M_1, M_2 = 0, \pm1, \pm2 \tag{49}$$

corresponding to the classical case.

In addition we also obtain in the q-deformed case terms proportional to the respective components of $\widehat{Q^2_m}$ $(m = m_1 + m_2 = 0, \pm1, \pm2)$, but with a coefficient containing a $(q - q^{-1})$ factor, which approaches zero when $q \to 1$. This is an important difference in the q-deformed case, which could lead to new results in physical applications.

After introducing the CGC in the tensor product (43) the explicit expression for the scalar operator is

$$S^0_0 = \frac{1}{\sqrt{[3]}}\{q^{-1}b^\dagger_1\tilde{b}_{-1} + qb^\dagger_{-1}\tilde{b}_1 - b^\dagger_0\tilde{b}_0\}. \tag{50}$$

We can consider the scalar operator obtained in this way as the image of the scalar operator L^0_0, defined in (21)

$$\sqrt{[2]}L^0_0 \to \sqrt{[3]}S^0_0. \tag{51}$$

The reason for this interpretation is that S^0_0 (50) commutes with the components (31) of the first-rank tensor I^1_m

$$[S^0_0, I^1_m] = [L^0_0, L^1_m] = 0 \qquad m = 0, \pm1. \tag{52}$$

Finally, we have to calculate the commutation relations of the scalar operator S^0_0 with the components of Q^2_m, for which we similarly find

$$[S^0_0, Q^2_m] = 0 \qquad m = 0, \pm1, \pm2. \tag{53}$$

Introducing the images of the operators L_0^0 and L_0^1 in equations (23), the images of q-deformations of the boson number operators can now both be obtained in terms of q-deformed vectors

$$[N_1]q^{N_2} \to \left(qb_1^\dagger\tilde{b}_1 - \frac{q^{-1}}{[2]}b_0^\dagger\tilde{b}_0\right)$$
$$[N_2]q^{-N_1} \to -\left(q^{-1}b_1^\dagger\tilde{b}_{-1} - \frac{q}{[2]}b_0^\dagger\tilde{b}_0\right). \tag{54}$$

An important difference with the constructions of [11, 12] is, therefore, that we are able to express the q-number operators (54) in terms of q-bosons and that we are able to construct a q-analogue of the physically relevant quadrupole operator. This difference underlines the important role of the conditions imposed for the evaluation of the commutation relations between q-vectors. The mapping procedure used by us for this purpose seems to be quite appropriate to obtain not only the simplest representations of q-deformed algebras, but also the embedding of the q-deformed $so(3)$ in $u(3)$, thereby answering some of the questions that remained open in [12].

5. Conclusions

In conclusion we have obtained the image of the subalgebra $so(3)$ of the tensor q-deformed algebra $sp(4, R) \sim o(3, 2)$ in terms of the q-deformed spherical tensors $b_m^\dagger$ and b_m of first rank and were able to exploit this image in the construction of a q-deformed $u(3)$ algebra. The key problem solved was to establish the commutation relations between these q-bosonic type operators. For this purpose a mapping procedure, developed here for the q-deformed case, was applied in which the operators $b_m^\dagger$ and b_m were used as the basic mapping tools. The images of the generators of the initial compact q-deformed subalgebra $su(2) \sim so(3)$ are obtained as tensor products of rank zero and one of the q-deformed vectors. The scalar operator is interpreted as a q-deformation of the number operator and the first-rank tensor as an angular momentum operator. The additional q-deformed quadrupole operator Q_m^2, which is considered as a q-analogue of the Elliott quadrupole operator [7], extends the initial $so(3)$ algebra to a q-deformation of $su(3)$. It must be emphasized once again, that working in terms of q-deformed spherical tensors provides a natural basis in the q-deformed case for obtaining the reduction of $u(3)$ to the angular momentum algebra $so(3)$.

From the calculated commutation relations between the scalar operator S_0^0 (43), the rescaled tensor operators $\widehat{I_m^1}$ (40) and $\widehat{Q_m^2}$ (49) and their respective limits when $q \to 1$, the following conclusions can be drawn. As a result of the commutation relations (41) the components of the operator $\widehat{I_m^1}$ close a q-deformation of an $su(2)$ algebra isomorphic to an $so(3)$ algebra, which we can interpret as the angular momentum algebra in this case. From the relations (52) it is obvious that the operators $\widehat{I_m^1}$ and S_0^0 generate a q-deformation of a $u(2)$ algebra and the following reduction is realized

$$u_{\tilde{q}}(2) \supset su_{\tilde{q}}(2) \oplus u_{\tilde{q}}(1). \tag{55}$$

The index $\tilde{q}$ denotes that this is not the standard q-algebra, but generally a q-deformation of the same type of 'classical' Lie algebra.

From the commutators (48) and (45) it can be concluded that the components of $\widehat{I_m^1}$ and $\widehat{Q_m^2}$, eight in total, close a q-deformation of an $su(3)$ algebra and with the scalar S_0^0 (53) a q-deformation of a $u(3)$ is generated. Moreover, the reduction chain

$$u_{\tilde{q}}(3) \supset su_{\tilde{q}}(3) \oplus u_{\tilde{q}}(1) \supset so_{\tilde{q}}(3) \oplus o_{\tilde{q}}(2) \tag{56}$$

important for applications in various nuclear structure models, is realized.

By stressing and exploiting analogies with a standard approach to algebraic structures in nuclear physics models, including the role of embedding relationships, we have obtained in a non-standard and novel way a representation of a q-deformed $u(3)$ algebra in terms of spherical tensors which have clear physical analogues in the classical algebraic models. This construction leads in a quite natural way to the reduction (56) which should pave the way for applications built on q-deformed quadrupole structures.

Acknowledgments

AIG is supported through contract No Φ-621 from the Bulgarian Ministry of Education and Science and HBG by grant No 2034459 from the South African Foundation for Research Development. These two authors also acknowledge support by the Institute for Nuclear Theory, University of Washington, where this work was initiated during programme INT-95-3.

References

[1] Biedenharn L C and Lohe M A 1995 *Quantum Group Symmetry and q-Tensor Algebras* (Singapore: World Scientific)
[2] Raychev P, Roussev R and Smirnov Yu F 1990 *J. Phys. G: Nucl. Part. Phys.* **16** L137
[3] Gupta R K, Cseh J, Ludu A, Greiner W and Scheid W 1992 *J. Phys. G: Nucl. Part. Phys.* **18** L73
[4] Bonatsos D, Faessler A, Raychev P P, Roussev R P and Smirnov Yu F 1992 *J. Phys. A: Math. Gen.* **25** L267
[5] Del Sol Mesa A, Loyola A, Moshinsky M and Velázquez V 1993 *J. Phys. A: Math. Gen.* **26** 1147
[6] Gupta R K and Ludu A 1993 *Phys. Rev.* C **48** 593
[7] Elliott J P 1958 *Proc. R. Soc.* A **245** 128
Elliott J P 1958 *Proc. R. Soc.* A **245** 526
[8] Draayer J 1993 Fermion models *Algebraic Approaches to Nuclear Structure* ed R Casten (London: Harwood Academic) p 423
[9] Rowe D J 1993 The nuclear symplectic model *Nuclear Structure Models: International Workshop (Oak Ridge, Tennessee, 16–25 March 1992)* ed R Bengtsson, J Draayer and W Nazarewicz (Singapore: World Scientific)
[10] Iachello F and Arima A 1987 *The Interacting Boson Model* (Cambridge: Cambridge University Press)
[11] Van der Jeugt J 1993 *J. Math. Phys.* **34** 1799
[12] Quesne C 1993 *Phys. Lett.* B **298** 344
Quesne C 1993 *Phys. Lett.* B **304** 81
[13] Smirnov Yu F, Tolstoi V and Kharitonov Yu 1991 *Sov. J. Nucl. Phys* **53** 593
[14] Georgieva A 1993 Tensorial structure of a q-deformed $sp(4, R)$ superalgebra *Preprint* IC/93/50 ICTP Trieste
[15] Georgieva A and Dankova Ts 1994 *J. Phys. A: Math. Gen.* **27** 1251
[16] Castanos O, Chacon E, Moshinsky M and Quesne C 1985 *J. Math. Phys.* **26** 2107
[17] Goshen S and Lipkin H J 1968 On the application of the group $Sp(4)$ or $R(5)$ to nuclear structure *Spectroscopic and Group Theoretical Methods in Physics* ed W A Friedman and H Feshbach (Amsterdam: North-Holland)
[18] Klein A and Marshalek E R 1991 *Rev. Mod. Phys.* **63** 375
[19] Asherova R M, Fursa D M, Georgieva A I and Smirnov Yu F 1993 *J. Phys. G: Nucl. Part. Phys.* **19** 1887
[20] Dobaczewski J, Geyer H B and Hahne F J W 1991 *Phys. Rev.* C **44** 1030
[21] Park P 1987 *Phys. Rev.* C **35** 805
[22] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[23] Menezes D P, Brink D M and Bonatsos D 1989 *Nucl. Phys.* A **494** 186
Menezes D P and Bonatsos D 1989 *Nucl. Phys.* A **499** 29